REPRINT

Time-Dependent Expressions for Delay, Stop Rate and Queue Length at Traffic Signals

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NOTE:

This report is related to the intersection analysis methodology used in the SIDRA INTERSECTION software. Since the publication of this report, many related aspects of the traffic model have been further developed in later versions of SIDRA INTERSECTION. Though some aspects of this report may be outdated, this reprint is provided as a record of important aspects of the SIDRA INTERSECTION software, and in order to promote software assessment and further research. This report was originally published by the Australian Road Research Board.

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TIME – DEPENDENT EXPRESSIONS FOR DELAY, STOP RATE AND QUEUE LENGTH AT TRAFFIC SIGNALS

by

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367 Traffic Signal Control Techniques

This reprint has been prepared by retyping the report including some minor revisions and reformatting.

Rahmi Akçelik

14 September 2011

REPORT SUMMARY

THE PURPOSE OF THIS REPORT

Is to present new expressions for delay, number of stops and queue length at traffic signals operating in a fixed-time mode so as to provide simple working models to predict traffic operating characteristics in both undersaturated (below capacity) and temporarily oversaturated (above capacity) conditions.

THIS REPORT SHOULD INTEREST

traffic engineers and researchers concerned with the design and operation of traffic signals.

THE MAJOR CONCLUSION OF THE REPORT ARE

(a)   the limitations of the existing formulae for delay, number of stops and queue length based on the steady-state (equilibrium) flow assumptions for undersaturated signals and the deterministic formulae for oversaturated signals can be overcome by using a co ordinate transformation technique; the use of the overflow queue concept provides a good basis for this purpose;

(b)   the new expressions predict finite values of delay, queue length and number of stops for flows near capacity with a smooth transition from below-capacity to above-capacity conditions; this is the contrast with the existing steady-stare expressions for undersaturated signals which predict infinite values for flows just below capacity and the deterministic expressions for oversaturated signals which predict zero (or very small) values for flows just above capacity; the difference is important since most signals operate under near-capacity conditions during peak periods of traffic.

AS A CONSEQUENCE OF THE WORK REPORTED, THE FOLLOWING ACTION IS RECOMMENDED

(a)   incorporate the new expressions (simplified forms) given in this paper in the relevant section of the revised Signalised Intersection Capacity and Timing Guide (to replace ARR 79);

(b)   extend to the method described to develop more comprehensive formulae to predict traffic operating characteristics in consecutive time periods;

(c)   study overflow queues at co-ordinated and vehicle actuated traffic signals; and

(d)   study the optimum signal timings derived using the new expressions in comparison with the timings derived using the traditional (Webster, Miller,etc.) expressions.

RELATED ARRB RESEARCH

P312   Effects of Traffic Patterns on Fuel Consumptions and Emission
P331   Analysis of SCATS
P347   Signalised Intersection Survey Method
FS1068   Australian Traffic Signalling
FS1094   Signalised Intersection Capacity and Timing Guide Revision

INFORMATION RETRIEVAL

ABSTRACT : This paper presents new approximate expressions for predicting delay, stop rate and queue length at traffic signals operating in a fixed-time mode for flow conditions which last for a finite period of time. The emphasis is on the use of the overflow queue concept which facilitates a better understanding of the transition from undersaturated conditions to oversaturated conditions, and allows for the derivation of simpler expressions which apply to both conditions. Firstly, deterministic expressions for delay, stop rate and queue length are derived for oversaturated conditions. Steady-state expressions for undersaturated signals are summarised. Then the limitations of steady-state and deterministic expressions for near-capacity conditions are discussed. Transition functions which overcome these limitations are given. These functions have been derived using the Whiting / Kimber-Hollis co-ordinate transformation technique. Each one of the new expressions for delay, stop rate and queue length is presented in a simple form consisting of a uniform component and an overflow component. The latter is an explicit function of the average overflow queue. Transition functions for the average overflow queues at isolated as well as co-ordinated signals are presented.

* Non-IRRD keywords

CONTENTS

1. INTRODUCTION

The normal analysis of congestion is as a steady-state phenomenon potentially in long-run equilibrium. ... For an intersection there might be a flow/delay relation which will relate an average equilibrium delay to a flow which will be expected to continue for some time without change. (This type of relation is) often shown to rise to infinite time as the rate of flow approaches some finite value. Clearly so long as the steady state assumption is retained, that the same rate of flow might reasonably be supposed to continue forever, this is just what would happen, but as flow approaches "capacity" this postulate becomes increasingly implausible. In such a situation the standing queue increases steadily in length and the time required to reach the steady state becomes longer, until on the extreme, there will be an infinite queue and it will take an infinite time to reach the steady state if flow is actually equal to or greater than "capacity". In fact (congestion) is always a peaked phenomenon lasting only for a limited length of time. Failure to take account of the temporary nature of (congestion) can lead to analytical problems.

(Neuburger 1971)

Various expressions for predicting the queue length, delay and stops at isolated, undersaturated signals were discussed in a paper by the author (Akçelik 1980). It was shown that, as in the case of delay and queue length, an expression to predict the stop rate (average number of stops per vehicle) could be derived using the overflow queue concept. Miller's (1968) approximate expression for the average overflow queue, i.e. the average number of vehicles left in the queue at the end of the green period, at isolated undersaturated signals, was recommended for use in the stop rate formula because of its explicit and direct formulation. It was also shown that an equivalent average overflow queue value could be calculated using the queue length prediction method given by Webster (1958). The importance of the average overflow queue parameter in various expressions for delay at undersaturated signals, and other discussions on the overflow queue concept can be found in Haight (1959), Miller (1964), Allsop (1972), McNeil and Weiss (1974), Ohno (1978) and Sosin (1980).

It appears that there has been an over-emphasis on some detailed aspects of the theoretical formulations of delay at undersaturated signals without due consideration to their practical relevance. As aptly stated by Allsop (1972):

'The appreciation of the relevance of these theoretical probabilistic arrival models was valuable in enabling probability theory to be applied to the analysis of the traffic signal queue. The mathematical fascination of such analyses has, however, caused them to be pursued to a level of intricacy far greater than is likely to find practical application. The behaviour of real traffic does not correspond closely enough to any of these very specific mathematical models to warrant, for practical purposes, the very refined solutions that have been obtained.

A major limitation of the expressions given in the papers referred to above is that they are valid for steady-state (equilibrium) conditions only. This means that the demand flow is, on average, less that the capacity available, i.e. undersaturated conditions apply. Following Miller's (1968) suggestion about the accuracy of his average overflow queue formula, the formulae given in Akçelik (1980) were considered to give good approximation for degrees of saturation (flow / capacity ratios) up to 0.95. The steady-state expressions are independent of the length of time period they are applied to. However, in oversaturated conditions, which last for a finite period of time, it becomes important that the analyses are based on the use of time-dependent flow and timing parameters.

There is now a growing amount of literature on temporarily oversaturated signals (e.g. Rahman 1973, Gazis 1974, Catling 1977, Yagar 1977, Branston 1978, Pignataro, et al. 1978, Mayne 1979, Michalopoulos 1979, Roberston 1979, Kimber and Hollis 1979, Luk and Sims 1979, Michalopoulos and Meyer 1980). Formulae given in some of these publications for predicting delays and queue lengths appear to be mathematically involved, especially those for consecutive time periods (e.g. see Mayne 1979). A simpler formulation for delay at isolated signals is adopted by Robertson (1979). This is based on a co-ordinate transformation (or, transition curve) technique which was conceived by Whiting for use in TRANSYT program and developed in detail for a variety of queuing situations by Kimber and Hollis (1979).

The purpose of this paper is to present new expressions for delay, stop rate and queue length which have been derived using the co-ordinate transformation technique. The emphasis in the new expressions is on the use of the overflow queue concept. This facilitates a better understanding of the transition from undersaturated (steady-state) conditions to (temporarily) oversaturated conditions, and allows for the derivation of simpler expressions which apply to both conditions.

The expressions given in this paper are intended for practical use in a way similar to the use of the traditional formulae by Webster (1958) and Miller (1968) which were limited to undersaturated cases. For this reason, the formulae given in this paper are based on the consideration of a single time period with zero initial queue.

A major contribution of this paper is considered to be presenting expressions to predict the stop rate at traffic signals. The importance of stops as a measure of performance has been discussed elsewhere (Akçelik 1980, 1981a).

The expressions for delay, stop rate and queue length are derived below for oversaturated conditions using a deterministic approach first. Steady-state expressions for undersaturated signals are summarised. Then the limitations of steady-state and deterministic expressions for near capacity conditions are considered and the co-ordinate transformation technique is utilised to obtain transition functions which are of use in both below and above capacity conditions. Simplified forms of these functions are presented for practical use. These expressions have been used in the new Australian Guide for Traffic signal capacity and Timing Analysis (Akçelik 1981b).

2. DETERMINISTIC EXPRESSIONS FOR OVERSATURATED SIGNALS

Methods to derived deterministic expressions for delay and queue length at oversaturated signals, i.e. when the average arrival flow rate exceeds capacity, are given in earlier publications by May (1965), May and Keller (1976), Neuburger (1971) and Gazis (1974). The basic model presented below assumes a constant arrival (demand) flow rate, q, which persists during a time interval of length, t. It also assumes, in the deterministic sense, that the queue length at the start of this time interval is zero. The capacity, $Q = s\ g\ /\ c$ (s = saturation flow, $g\ /\ c$ = effective green time / cycle time ratio), which is constant as determined by the fixed-time operation of signals, is exceeded by the arrival flow rate, q, by an amount equal to $z\ Q = q - Q$ (where $z = x - 1$, $x = q\ /\ Q$ is the degree of saturation, i.e. flow / capacity ratio). Thus, the oversaturation queue is assumed to increase linearly from zero to a maximum value of $(z\ Q\ t)$ at the end of the time interval, t. Provided the arrival flow rate in the subsequent interval is less than the capacity $(q_2 < Q)$, the oversaturated queue will be cleared at time $t_2 = (z\ /\ z_2)\ t$ after the end of interval t, where $z_2 = 1 - x_2 > 0$ and $x_2 = q_2\ /\ Q < 1.0$.

In the derivations below, $q_2 = 0$, and hence $t_2 = z\ t = (x - 1)\ t$, is used so as to exclude the delays and stops to vehicles which arrive during the subsequent period.

The basic model is illustrated in *Figure 1*. The cumulative numbers of arrivals and departures are shown in the top part of *Figure 1* for the following example:

> cycle time, c = 120 s, effective green time, g = 30 s,
> arrival flow, q = 360 veh/h, arrivals per cycle, q c = 12 veh, saturation flow,
> s = 1200 veh/h, departures per cycle, s g = 10 veh, capacity,
> Q =300 veh/h, degree of saturation, x = 1.2, z = x - 1 = 0.2, and
> excess flow, z Q = q - Q = 60 veh/h (q c - sg = 2 veh per cycle)

A time interval of t = 10 min (5 cycles) is considered. The queue at the end of interval t is z Q t = 10 veh, and the time to clear this queue is t_2 = z t = 0.2 $_x$ 10 = 2 min (1 cycle).

The following can be seen in the top part of *Figure 1*:

(a) The area between the cumulative arrival curve (linear with slope q) and departure curve (sawtooth with zero slope during the red period and slope s during the green period) represents the total delay. The horizontal distance between the two curves represents the delay to an individual vehicle (e.g. AA' = 116 s for vehicle 23).

(b) The vertical distance between the two curves represents the queue length (number of vehicles in queue) at a given time (e.g. BB'). The overflow queue is the queue at the end of a signal cycle (e.g. CC' = 8 veh at the end of the fourth cycle).
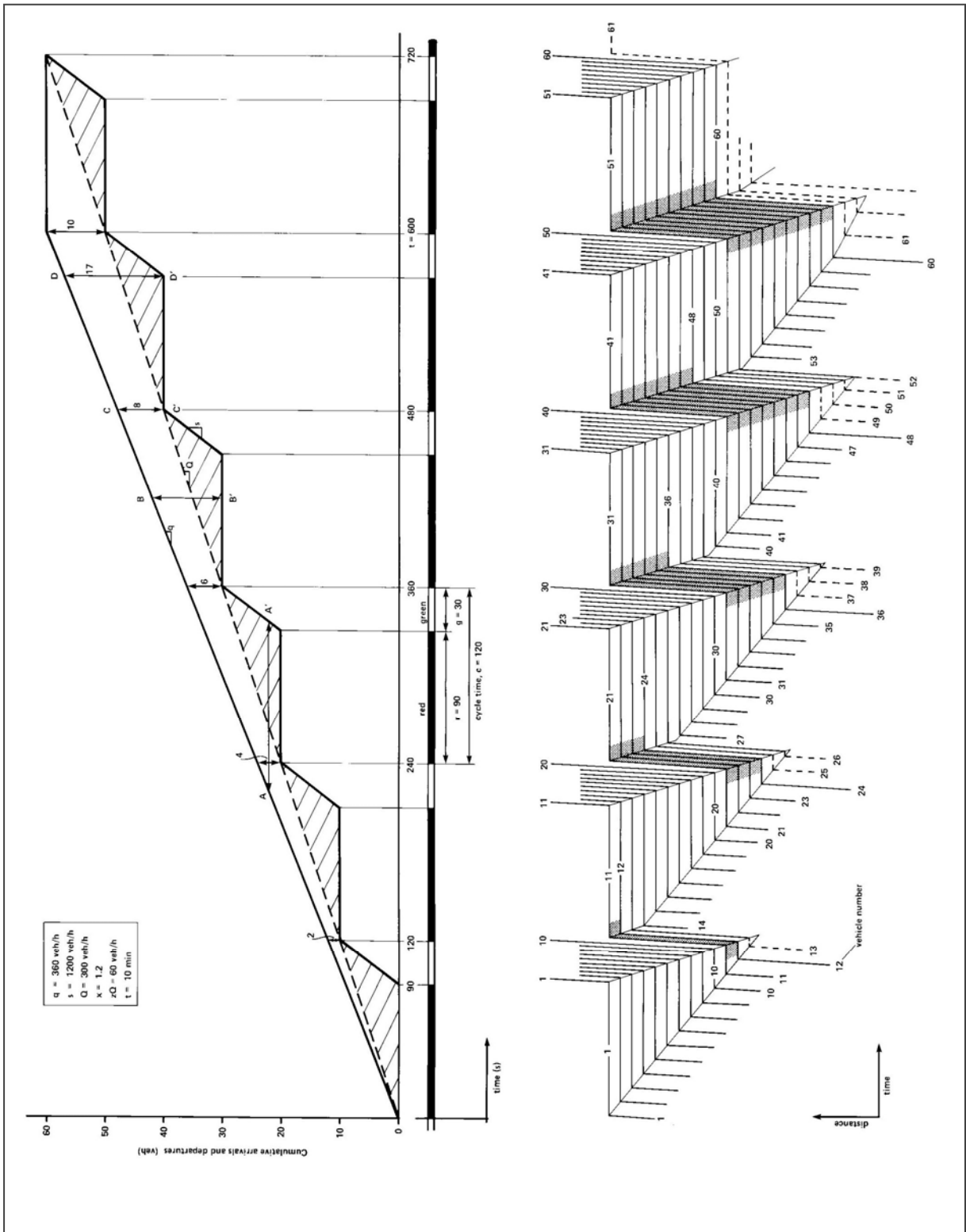
Figure 1 - Deterministic model of oversaturated signals (an example)

The idealised time-distance trajectories of individual vehicles are shown in the bottom part of *Figure 1*. The assumption of the uniform arrivals (constant headways, 1 / q) is used as in the case of deriving the *uniform component* of delay, stops and queue length at undersaturated signals (Akçelik 1980). The relationship between the overflow queue and multiple stops, i.e. more than one stop per vehicle, can be observed from *Figure 1*. For example, an overflow queue of 2 vehicles at the end of the first cycle corresponds to vehicles 11 and 12 which make two stops before clearing the intersection. Similarly, an overflow queue of 4 vehicles at the end of the second cycle corresponds to vehicles 21 to 24 which are stopped twice, and so on (initial stops by vehicles 13 and 25, 26 in the first two cycles are neglected as explained below).

General expressions for the overflow queue, delay and stops in an oversaturated cycle are given below using the parameters shown in *Figure 2*.

Overflow queue at the end of the *ith* cycle:

$$n_i = n_{i-1} + q c + s g \tag{1}$$

Total delay during the *ith* cycle:

$$D_i = n_{i-1} c + 0.5 (q c^2 + s g^2) \tag{2}$$

Total number of stops during the ith cycle:

$$H_i = n_{i-1} + q c \tag{3}$$

Theoretically, the number of stops per cycle is $(n_{i-1} + q r) / (1 - y)$, where $y = q / s$ (flow ratio). *Equation (3)* gives a smaller number of stops because it neglects the stops made by vehicles which arrive at the back of a moving queue from the previous cycle, i.e. only the stops by (q c) vehicles arriving in a cycle are accounted for. For example, the initial stops by vehicle 13 in the second cycle, vehicles 25 and 26 in the third cycle, vehicles 37 to 39 in the fourth cycle, etc., are neglected (shown by dotted lines in the bottom part of *Figure 1*). This assumption was made so as to compensate for drivers who may adjust their speed to avoid a stop when faced with a red signal and moving queue ahead.
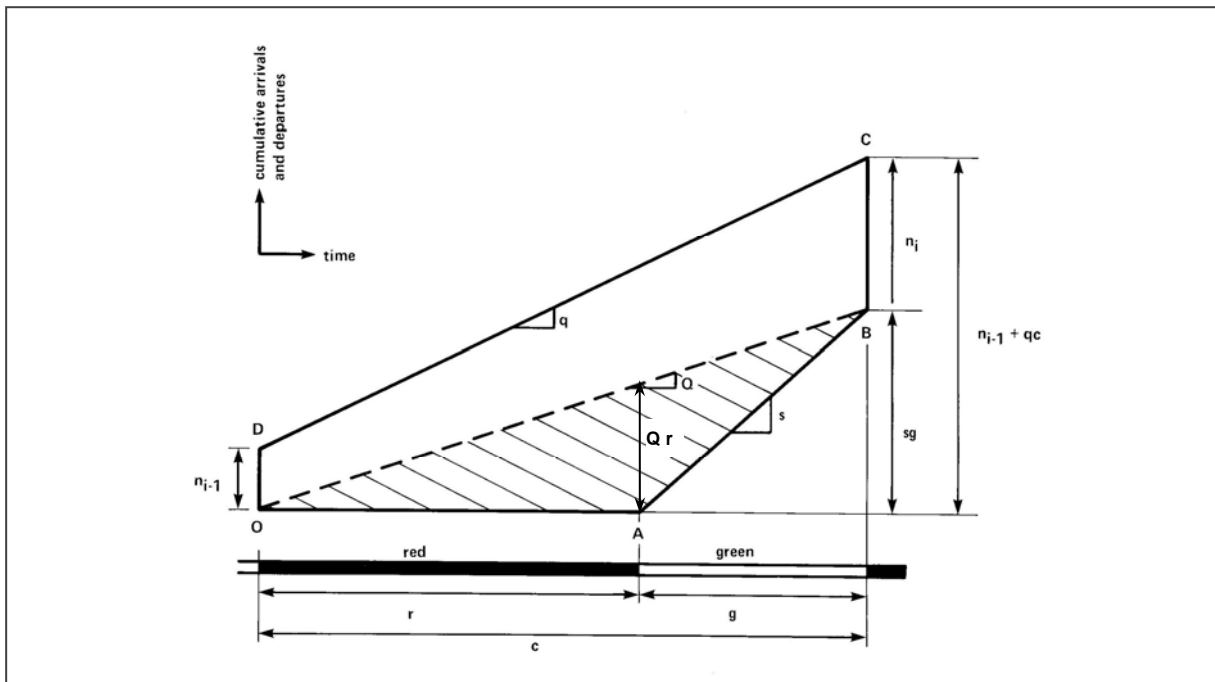

Figure 2 - An oversaturated signal cycle

The following deterministic expressions (subscript d is used to indicate this) for the average overflow queue, total delay and stop rate as experienced by (q t) vehicles during the period $t + t_2 = x\, t$ (in x t / c cycles) are derived from *Equations (1) to (3)*.

Average overflow queue is given by:

$$N_d = 0.5\,(q - Q)\,t = 0.5\,z\,Q\,t \qquad\qquad (4)$$

where

N_d = average overflow queue in vehicles total number of vehicles queued in all lanes),

q = arrival flow which persists during the time interval t (q in vehicles per hour, t in hours), and

Q = capacity (in vehicles per hour).

Total delay is given by:

$$D_d = 0.5\,q\,r + N_d\,x \qquad\qquad (5)$$

where

D_d = total delay (in vehicle-hours per hour, or simply 'vehicles'), and

q r = total number of vehicles which arrive during the red period (q = flow in vehicles per second, r = effective red time in seconds)

The average delay per vehicle can be calculated from d = D / q where q is in vehicles per second.

Stop rate is given by:

$$h_d = 1 + N_d / (s\,g) \qquad\qquad (6)$$

where

h_d = stop rate (average number of stops per vehicle), and

s g = total number of departures (capacity) per cycle (s = saturation flow in vehicles per second, g = effective green time in seconds, s g in vehicles).

The total number of stops can be calculated from H = h q, where H is in the same units as q.

Also of interest is the average queue length at the start of the green period. This is given by:

$$N_{dr} = Q\,r + N_d \qquad\qquad (7)$$

where Q, r, N_d are as in the above expressions. The maximum (stationary) queue length occurs in the last cycle during the time interval t (DD' in *Figure 1*), and is given by $N_c = 2\,N_d + (s - q)\,g$.

It is seen that *Equations (5), (6) and (7)* consist of two components:

(a)   A *uniform* component which corresponds to the triangular area OAB in *Figure* 2 (which is seen to be a fixed amount in each cycle in *Figure 1*). This is the at-capacity value of the statistic concerned, and can be obtained from the uniform component of the expression for under-saturated signals (*Equations 9 to 11* given below) by putting x = 1 (u = y, or q = Q).

(b)   A deterministic *overflow* component which corresponds to the area OBCD in *Figure 2*, and is expressed as a function of the average overflow queue given by *Equation (4)*. It can be shown that the overflow components of average delay ($d_o = N_o\,x / q$) and stop rate ($h_o = N_o / (s\,g)$) are related as ($h_o = d_o / c$) where c is the cycle time.

It should be noted that z > 0 and x > 1 in *Equations (4) to (7)*. It is also important to note that N_d, and hence the overflow component of each expression, is a linear function of the time interval, t.

The following values are calculated from above expressions for the example shown in *Figure 1* (using t = 1 / 6 h):

　　　average overflow queue, N_d = 5.0 veh,
　　　total delay, D_d = 10.5 veh-h/h,
　　　average delay, d_d = 105.0 sec,
　　　stop rate, h_d = 1.5 stops/veh
　　　total number of stops, H_d = 540 stops per hour,
　　　average queue at the start of green, N_{dr} = 12.5 veh, and
　　　maximum queue, N_c = 17.0 veh.

3. STEADY-STATE EXPRESSIONS FOR UNDERSATURATED SIGNALS

It was emphasised in an earlier publication (Akçelik 1980) that the expressions for delay, stop rate and queue length at undersaturated signals (i.e. when the average arrival flow rate is below capacity) are interrelated, and that each could be considered to have a *uniform* component and a *random* component. The random component was expressed as a function of average overflow queue.

At undersaturated signals the overflow queues are due to random variations in arrival flow rates which result in some cycles being oversaturated. However, because the arrival flow rate is, on average, below capacity, these queues will be discharged in subsequent cycles. The average overflow queue parameter represents the steady-state assumption which is the basis of the delay, stop rate and queue length expressions given in Akçelik (1980). These are summarised below (subscript s is used to denote steady-state).

Miller's (1968) formula for predicting the average overflow queue at undersaturated fixed-time signals is:

$$N_s = 0.5 \{exp \, [-1.33 \, \text{sqrt} \, (s \, g) \, (1 - x) / x]\} / (1 - x) \tag{8}$$

It is interesting to note that the numerical values of the average overflow queue given by Wormleighton (1965) are very close to those given by *Equation (8)*.

The Miller (1968) formula for delay is based on an explicit use of the average overflow queue parameter, N_s, given by *Equation (8)*

$$D_s = 0.5 \, q \, c \, (1 - u)^2 / (1 - y) + [(1 - u) / (1 - y)] \, N_s \tag{9}$$

where

u   =   green time ratio (effective green time /cycle time, g/c), and

y   =   flow ratio (flow/saturated flow, q/s).

Although a more complicated expression is possible as proposed by Ohno (1978), the accuracy of *Equation (9)* is considered to be sufficient for all practical purposes (see the Appendix).

The steady-state expression for stop rate is:

$$h_s = f \, [(1 - u) / (1 - y) + \, N_s / (q \, c)] \tag{10}$$

where f is a correction factor to allow for partial stops (slowdowns). A general purpose value of f = 0.9 was recommended in Akçelik (1980).

The average queue length at the start of green period is given by:

$$N_{sr} = q \, r + \, N_s \tag{11}$$

where N_s is given by *Equation (8)*, and (q r) is the number of vehicles in queue at the start of green period if uniform arrivals were assumed (q = average arrival flow rate in vehicles per second, r = effective red time in seconds).

4. TRANSITION FUNCTIONS

Let us now consider how the two sets of expressions given above, namely, *Equations (4) to (7)* for *oversaturated* conditions, and *Equations (8) to (11)* for *undersaturated* conditions, can be related together. The limitations of these two sets of expressions in predicting traffic operating characteristics for near-capacity conditions is discussed below in terms of the average overflow queue expressions. The same considerations apply to the delay, stop rate and queue length expressions.

As seen in *Figure 3*, which shows that average overflow queues predicted by *Equations* (4) and (8), for the numerical example given by Robertson (1979), a zero overflow queue value is predicted by the deterministic expression for a flow rate equal to the capacity whereas the steady-state (probabilistic) expression predicts infinite overflow queue values for flows just below capacity. This indicates clearly that neither set of equations given in the preceding sections can be expected to give reasonable results for flows near capacity.
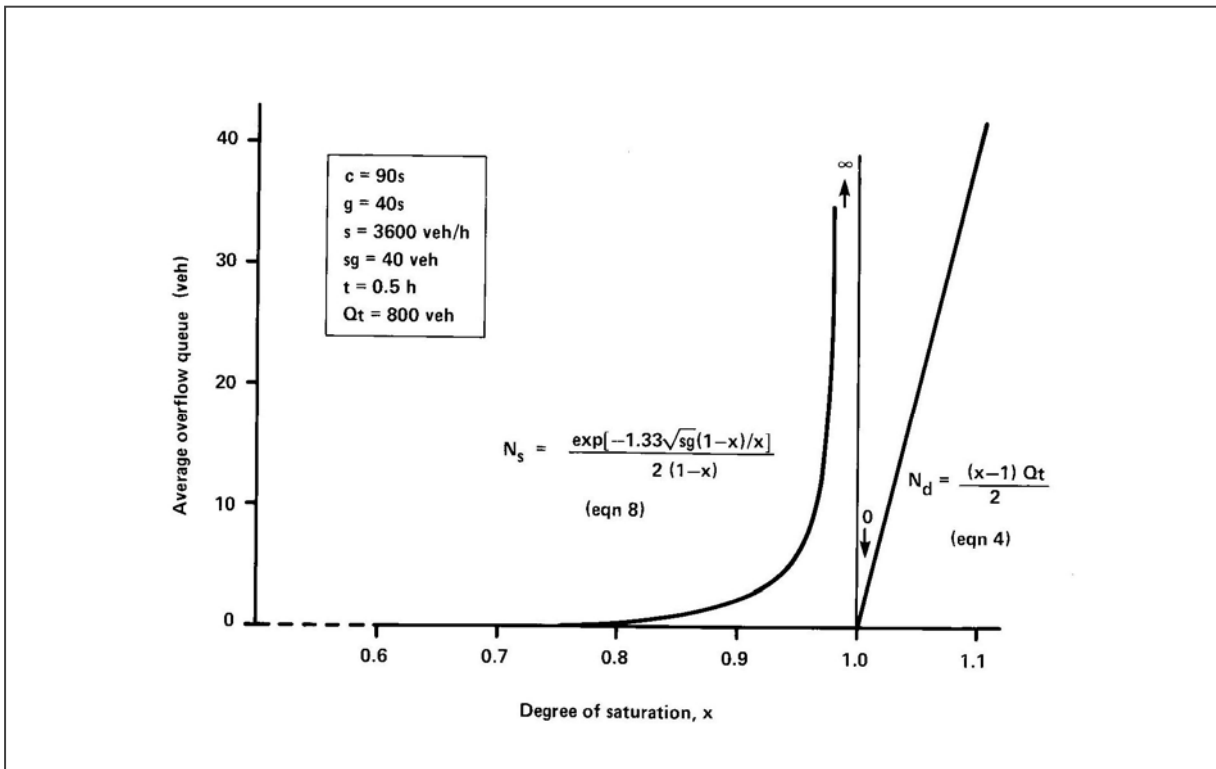
Figure 3 - Prediction of the average overflow queue by steady-state (x < 1) and deterministic (x > 1) expressions
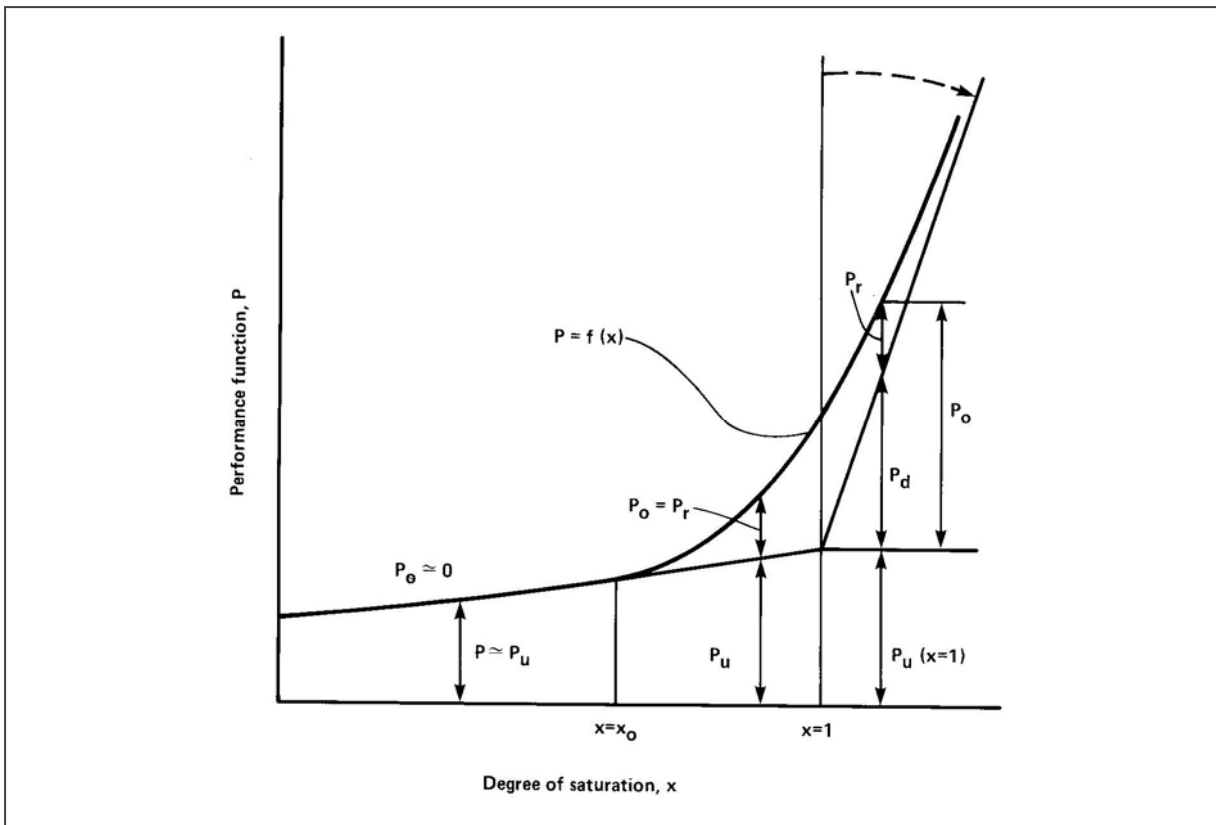

Figure 4 - Uniform, random and overflow components of a general performance function

The problem is solved using the co-ordinate transformation technique described in detail by Kimber and Hollis (1979). The technique was originally conceived by P.D. Whiting for deriving a 'random delay' expression for the TRANSYT computer program (Robertson and Gower 1977, Robertson 1979, Robertson, et al 1980). This technique transforms the steady-state function to a *transition function* which has the line representing the deterministic function as its asymptote. The result is not only to obtain realistic finite values of the performance function concerned for flows around capacity, but also to add a random component to the deterministic oversaturation function.

Figure 4 shows a general performance function, P, which represents any one of the delay (total or average, D, d) , stops (rate or total number, h, H), or the average queue length (N) functions. This can be expressed as:

$$P = P_u + P_o \tag{12}$$

where

P_u =   uniform component, and

P_o =   overflow component.

The uniform component, P_u, corresponds to the first terms of *Equations (5) to (7)* for x > 1, and *Equations (9) to (11)* for x < 1. As shown in *Figure 4*, the uniform component is constant for x > 1, and is equal to the value at capacity, P_u (x = 1).

The overflow component can be expressed as a function of the average overflow queue, $P_o = f(N_o)$, as in the second terms of *Equations (5) to (7)* and *Equations (9) to (11)*. The value of N_o can be predicted by a transition function as explained below. As shown in *Figure 4*, for flows above capacity (x > 1), the overflow component, P_o, is the sum of a deterministic component, P_d, and a random component, P_r. For flows below capacity (x < 1), the overflow component is equivalent to the random component, $P_o = P_r$ since P_d =0. It is also seen in *Figure 4* that the overflow component is zero below a certain degree of saturation (x < x_o) since $N_o \cong 0$ as predicted by *Equation (8)*.

4.1 AVERAGE OVERFLOW QUEUE

In order to facilitate the derivation of a transition function for the average overflow queue, the following expression has been developed by the author for undersaturated signals, which is a simple approximation to Miller's steady-state expression (*Equation 8*):

$$N_s = 1.5 (x - x_o) / (1 - x) \qquad \textit{for } x > x_o \tag{13}$$
$$= 0 \qquad \qquad \qquad x \leq x_o$$

where x_o is a degree of saturation below which the average overflow queue is approximately zero, and is given by:

$$x_o = 0.67 + s\, g / 600 \tag{13a}$$

where (s g) is in vehicles.

Two other steady-state expressions for the average overflow queue at undersaturated signals are of interest. McNeil's (1968) *upper-bound* expression (for Poisson arrivals):

$$N_s = 0.5 / (1 - x) \tag{14}$$

and Miller's (1963) approximate expression:

$$N_s = (x - 0.5) / (1 - x) \qquad \textit{for } x > 0.5 \tag{15}$$
$$= 0 \qquad \qquad \qquad x \leq 0.5$$

This is also considered to be an *upper-bound* expression, it is simply 0.5 less than the value given by *Equation (14)* for x > 0.5.
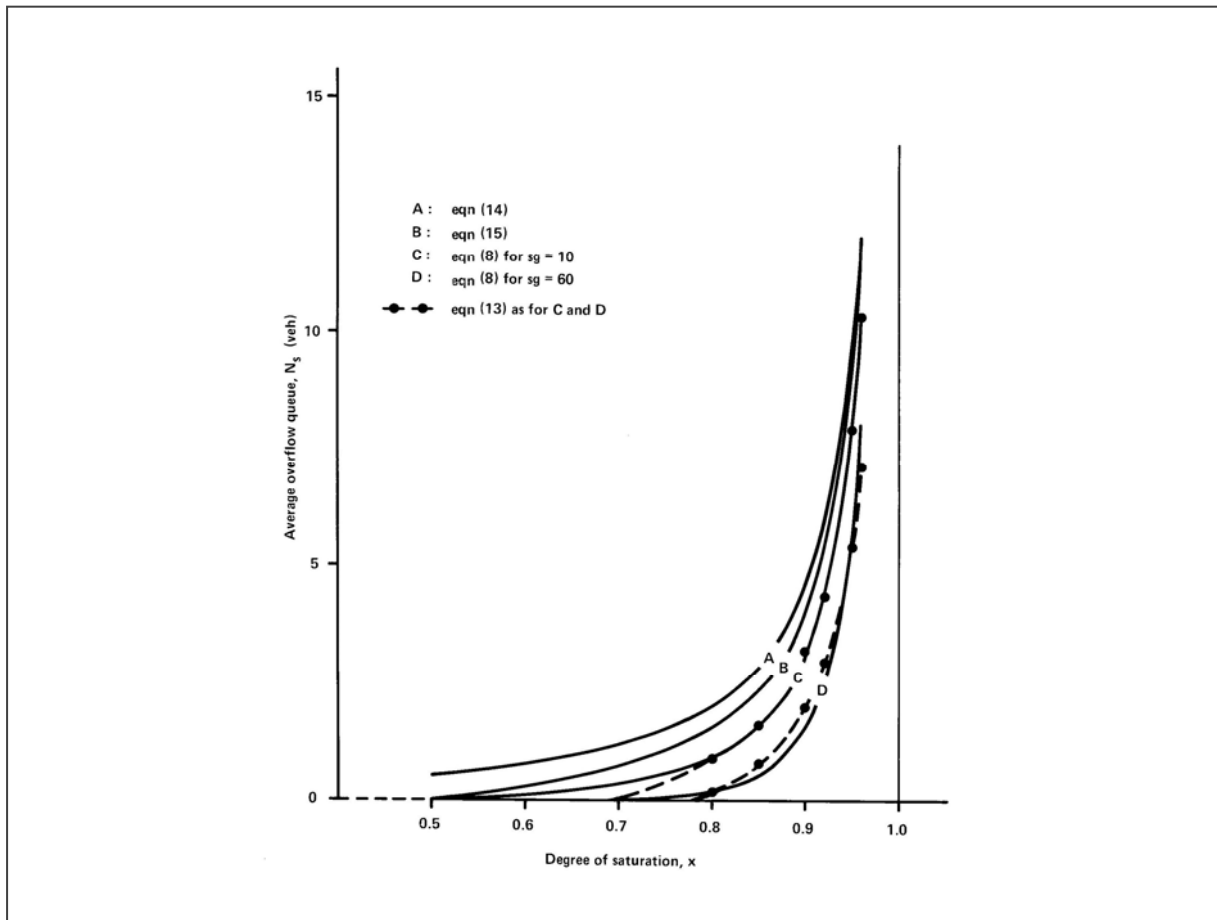
Figure 5 - Average overflow queues predicted by various steady-state (x < 1) expressions

The overflow queues for undersaturated conditions predicted by *Equations (8), (13), (14), and (15)* are shown in *Figure 5*. It is important to note that:

(a) *Equation (14)* predicts non-zero overflow queues for all degrees of saturation, whereas the other three expressions predict zero overflow queues for degrees of saturation below a certain value, and

(b) *Equations (14) and (15)* are not dependent on the (s g) parameter which is an indicator of the relative importance of various movements at an intersection (high s g values for major movements and low sg values for minor movements).

It appears that these two effects combined may have some impact on signal timing optimisation when there are marked differences between major and minor movement characteristics.

The transition function for average overflow queue which is obtained using the co-ordinate transformation technique to relate *Equation (13) to Equation (4)* is:

$$N_o = 0.25 \, Q \, t \, [z + sqrt \, (z^2 + 12 \, (x - x_o) / (Q \, t))] \qquad \text{for } x > x_o \qquad (16)$$
$$= 0 \qquad\qquad\qquad\qquad\qquad x \leq x_o$$

where

$Q \, t$ = throughput, i.e. the maximum number of vehicles which can be discharged during the time interval, t (Q is the capacity in vehicles per hour, t in hours),

z = $x - 1$ ($x = q / Q$, degree of saturation, q is the average arrival flow rate which persists during the time interval, t),

x_o = $0.67 + s \, g / 600$ as in *Equation (13)*.

It should be noted, for the correct use of *Equation (16)*, that the parameter z has a negative value for flows below capacity (x < 1).

It should also be noted that the derivation of *Equation (16)*, as well as other transition functions given in this section, involve some minor simplifying assumptions.

The transition function for average overflow queue based on *Equation (14)* is:

$$N_o = 0.25\ Q\ t\ [z + sqrt\ (z^2 + 4\ x\ /\ (Q\ t))] \qquad (17)$$

This expression is of interest because of its relevance to the delay function used in the TRANSYT computer program (Robertson and Gower 1977). As in *Equation (14)*, this function predicts non-zero overflow queues for all degrees of saturation, and can be considered to be an *upper-bound* expression.

The difference between the average overflow queue functions obtained from *Equations (16) and (17)* can be seen in *Figure* 6 which illustrates examples for s g = 10 and s g = 80 when Q t = 30 s g (e.g. for t = 1 h, c = 120 s).

4.2 TOTAL DELAY

To derive a transition function for total delay, let us replace random delay term of *Equation (9)* by the following approximate term:

$$D_r = ((1 - u)\ /\ (1 - y))\ N_s \cong N_s\ x \qquad (18)$$

With this approximation, if N_o is predicted by the *upper-bound* expression, *Equation (14)*, the random delay is:

$$D_r = 0.5\ x^2\ /\ (1 - x) \qquad (19)$$

This is the second term of Webster's (1958) delay formula.

Since *Equation (18)* is the same as second term of *Equation (5)*, the transition function is also of the same form. Therefore, total delay can be expressed as:

$$D = D_u + D_o \qquad (20)$$

where the uniform delay component is:

$$D_u = 0.5\ q\ c\ (1 - u)^2\ /\ (1 - y) \qquad \textit{for } x < 1 \qquad (20a)$$
$$= 0.5\ q\ r \qquad x \geq 1$$

and the overflow component is:

$$D_o = N_o\ x \qquad (20b)$$

where N_o is given by *Equation (16)*.

If N_o given by *Equation (17)* is used, D_o corresponds to the transition function given by Robertson (1979) which was obtained directly from *Equation (19)*. In this case, *Equation (20)* can be considered to be an *upper-bound* expression for total delay.

It should be noted that the approximation introduced in *Equation (18)* is relevant to degrees of saturation, x, in the range between about 0.65 and 0.95 because N_o is zero below about 0.65 (or x_o in *Equation (16)*) and the steady-state expressions are valid for x up to 0.95. The effect of this approximation is considered to be negligible for most practical purposes (see the Appendix).
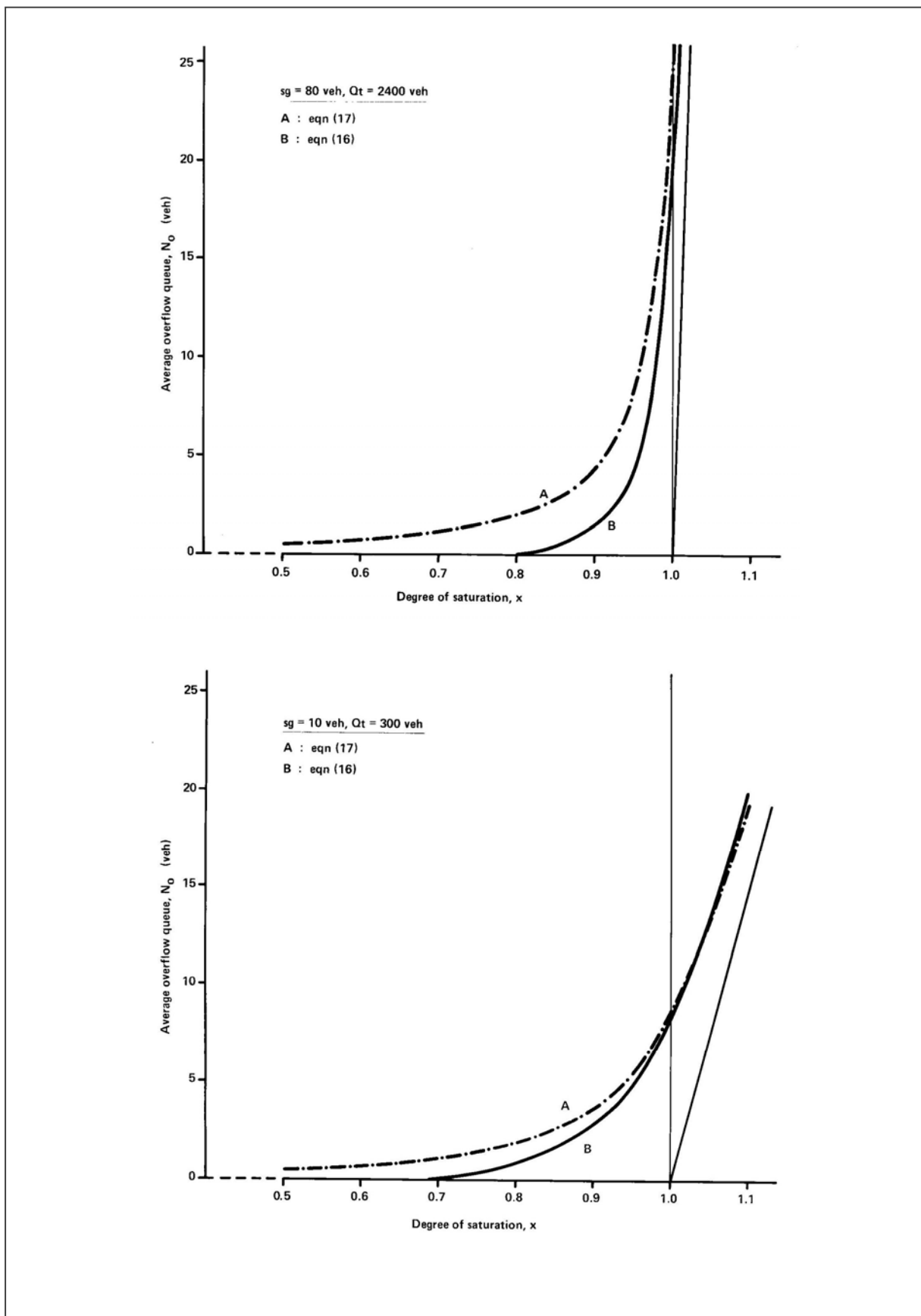
Figure 6 - Average overflow queues predicted by two transition functions

4.3 STOP RATE

The following transition function has been derived for the overflow component of stop rate by relating the second terms of *Equations* (10) and (6):

$$h_o = 0.25 \, (Q \, t \, / \, (s \, g)) \, [z + sqrt \, (z^2 + 12 \, (x - x_o) \, / \, (q \, t))] \qquad for \; x > x_o \qquad (21)$$
$$= 0 \qquad\qquad\qquad\qquad x \leq x_o$$

Therefore, the stop rate function can be expressed as:

$$h = f \, (h_u + h_o) \qquad\qquad (22)$$

where the uniform component is:

$$h_u = (1 - u) \, / \, (1 - y) \qquad\qquad for \; x < 1 \qquad (22a)$$
$$= 1.0 \qquad\qquad\qquad x \geq 1$$

the overflow component h_o is given by *Equation (21)*, and f is the correction factor for partial stops as in *Equation (10)*.

The overflow component of stops rate based on the use of N_o calculated from *Equation (17)*, is the same as *Equation (21)* except that 12 $(x - x_o)$ under the square root must be replaced by (4 x). If used in *Equation (22)*, this could be considered to give an *upper-bound* expression for stop rate.

4.4 AVERAGE QUEUE LENGTH AT THE START OF THE GREEN PERIOD

The average number of vehicles in the queue at the start of the green period (at the end of red period) is given by:

$$N_r = N_u + N_o \qquad\qquad (23)$$

where the uniform component is

$$N_u = q \, r \qquad\qquad for \; x < 1 \qquad (24)$$
$$= Q \, r \qquad\qquad x \geq 1$$

and the overflow component, N_o is simply the average overflow queue given by *Equation (16)*, or *Equation (17)*.

Equation (23) gives the maximum stationary stop-line queue in an average cycle. The maximum queue in any cycle can be calculated roughly as twice as the value of N.

The maximum back of the queue occurs sometime after the start of green as seen at the bottom part of *Figure 1*. The average value of this queue can be estimated from

$$N_m = (q \, r \, / \, (1 - y)) + N_o \qquad\qquad (25)$$

See Equation (15) in Akçelik (1980).

N_m is the physical end of the queue as perceived by the drivers. While the back of the queue increases from N_r at the start of green period (*Equation 23*) to N_m at some time after the start of green period, the front of the queue will be moving forward. At the time when N_m is reached, all vehicles will be moving, some beyond the stop line, i.e. will have already cleared the intersection.

5. SIMPLIFIED EXPRESSIONS

Equations (20) to (23) may appear to be a little cumbersome for use in practice. The condition of a fixed value of the uniform component for x > 1 in these expressions can be relaxed at the expense of obtaining slightly higher values of the performance function for oversaturated conditions. In the case of the stop rate expression a further simplification is introduced which compensates for this effect. The following expressions are recommended for practical use:

The total delay:

$$D = 0.5 \, q \, c \, (1 - u)^2 / (1 - y) + N_o \, x \qquad\qquad (26)$$

The stop rate:

$$h = f \, ((1 - u) / (1 - y) + N_o / (q \, c \,)) \qquad\qquad (27)$$

The average (stop-line) queue at the start of the green period:

$$N_r = q \, r + N_o \qquad\qquad (28)$$

It is seen that these expressions have the same form as the steady-state expressions for undersaturated conditions (*Equations 9 to 11*), except for the second term *Equation (26)* due to the approximation given by *Equation (18)*. Of course, the difference is that the average overflow queue, N_o, which determines the overflow components of the above expressions, is to be calculated from the transition function given by *Equation (16)*, or alternatively from *Equation (17)* for an *upper-bound* expression.

Equation (27) provides a convenient method of estimating stop rate at signals since separate calculation of an overflow component, h_o, from *Equation (21)* is avoided.

Figures 7 to 9 illustrate the average delay (d = D / q), stop rate, h, and average queue at the start of the green period, N_r, obtained from *Equations (20) to (23)*, with N_o calculated from *Equation (16)*, as a function of the degree of saturation, x. These diagrams are for the data used in an example by Robertson (1979). The overflow components of each function (curve T) is shown separately at the top part of each diagram together with the corresponding steady-state (curve S) and deterministic oversaturation (line D) functions. The stops shown in *Figure* 8 are uncorrected for partial stops, i.e. not multiplied by factor f.

The values of delay, stop rate and queue length obtained from simplified expressions (*Equations 26 to 28)* are shown in *Figures 7 to 9* as dots in the bottom part of each diagram for the cases where there is a difference from those given by respective *Equations (20) to (23)*.
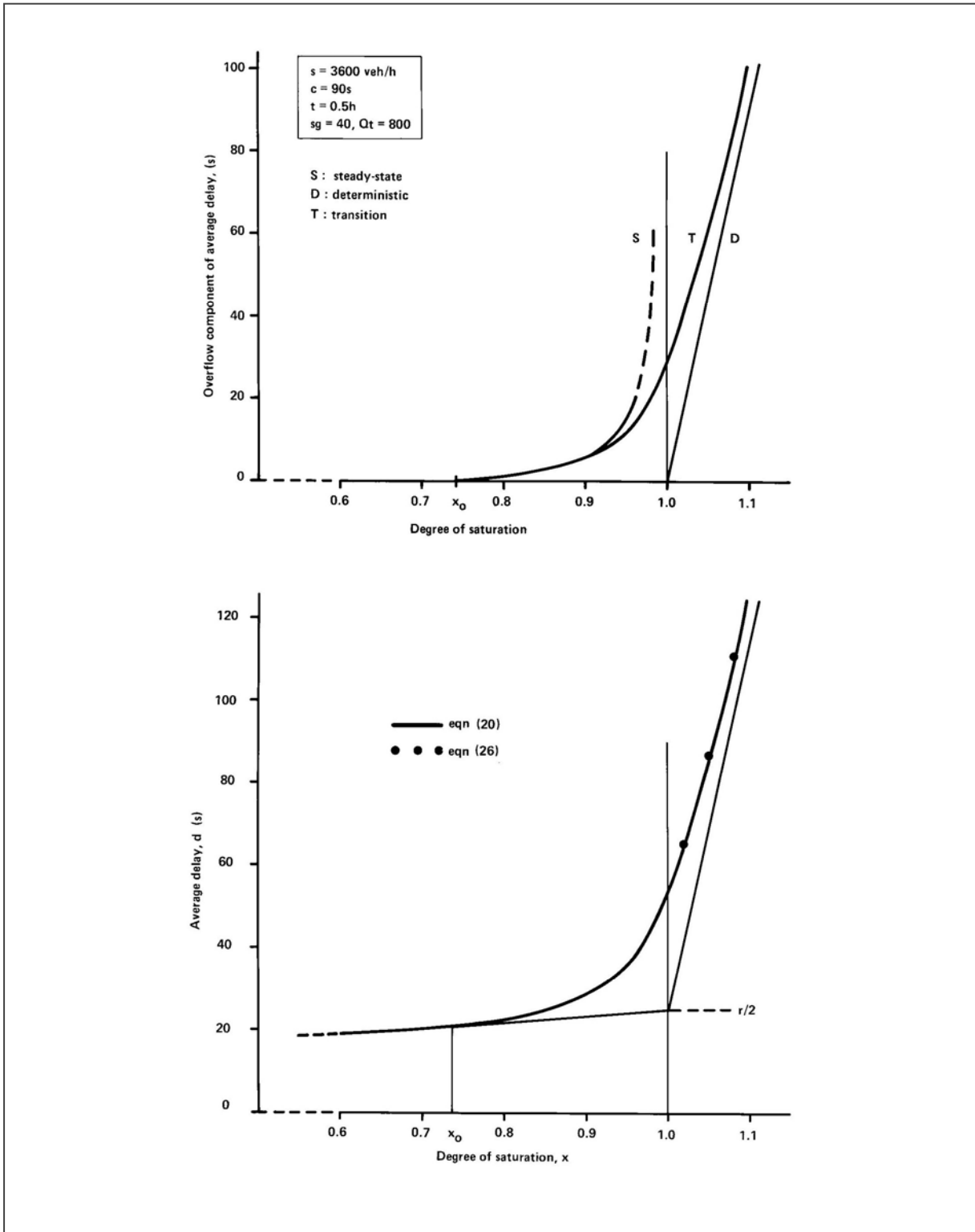
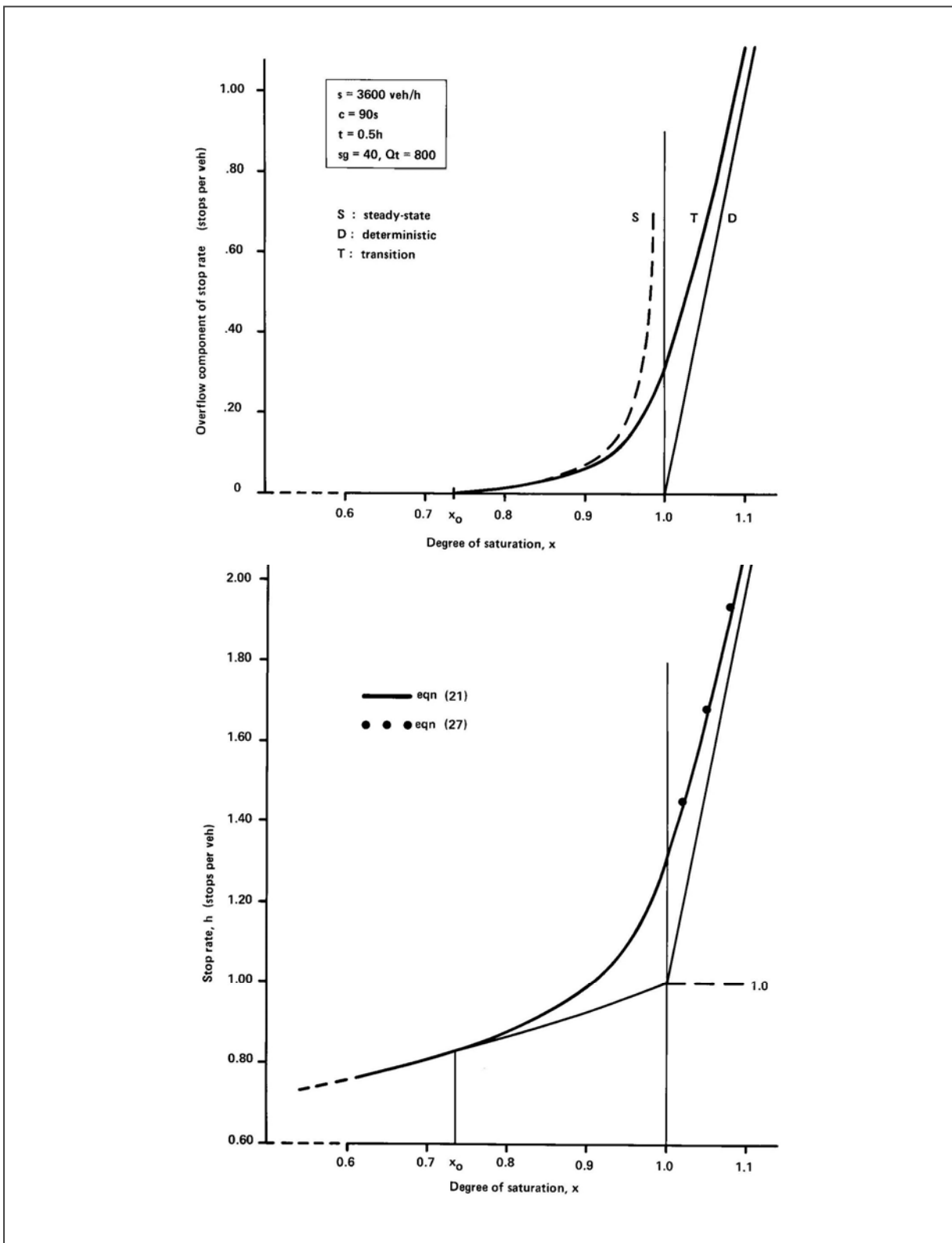*Figure 7 - Average delay as a function of the degree of saturation*

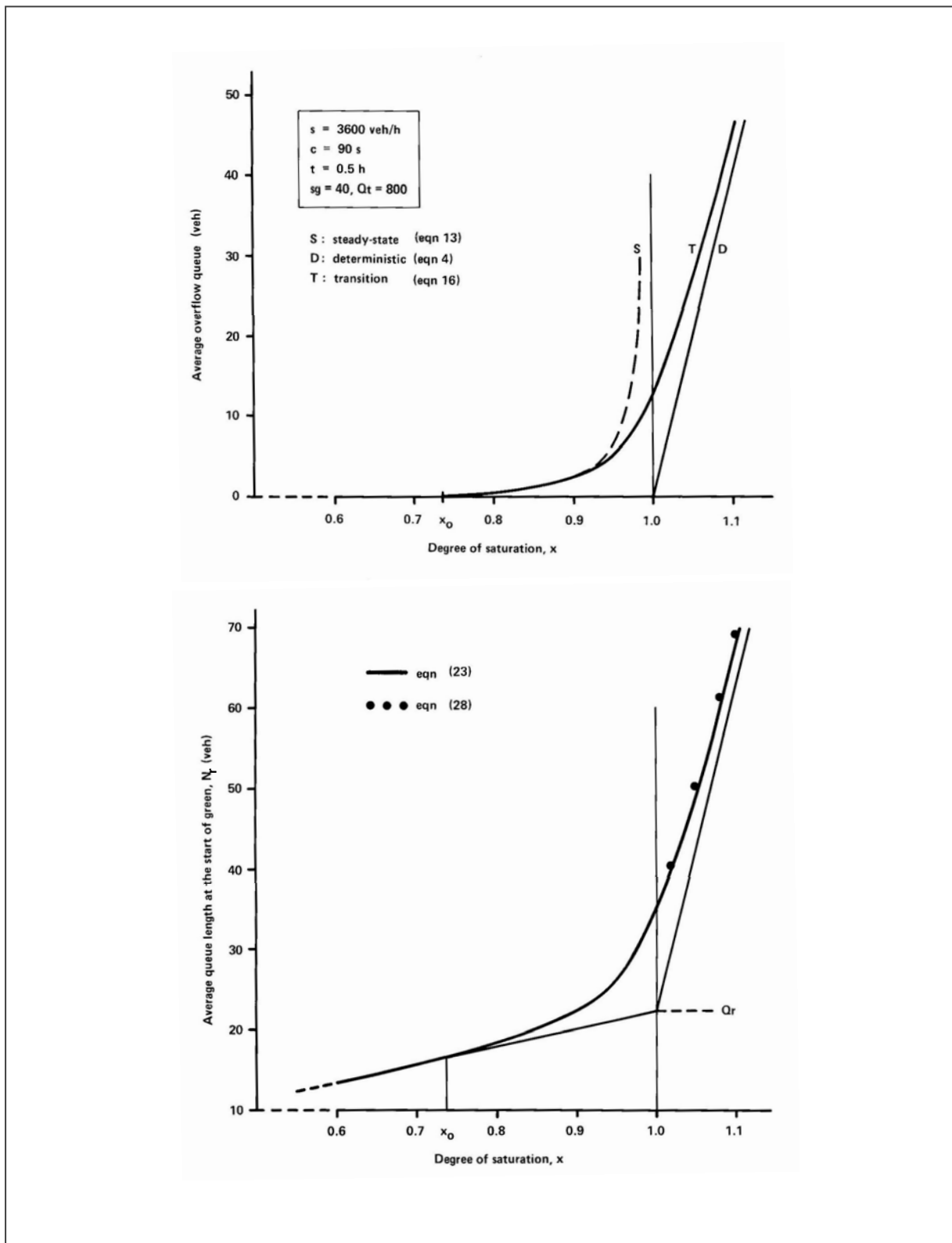*Figure 8 - Stop rate as a function of the degree of saturation*

Figure 9 - Average queue length at the start of green period as a function of the degree of saturation

6. CO-ORDINATED SIGNALS

It is interesting to note that a transition function was first used for co-ordinated signals rather than isolated signals (Robertson and Gower 1977, Robertson et al. 1980). The evidence given in Robertson (1969, App. 2) indicates that the effect of random variations in flow levels from cycle to cycle is smaller at co-ordinated signals that at isolated signals, probably because of the structured nature of arrival and departure patterns. In a way similar to the assumption made by Robertson (1969) for deriving a *random delay* term for the TRANSYT program, it may be assumed that the average overflow queue at undersaturated co-ordinated signals is half the average overflow queue at undersaturated isolated signals with the same x and (s g) values. Although research is needed to substantiate this, the assumption provides a reasonable working model in the meantime.

The transition function for average overflow queue at co-ordinated signals is then given by:

$$N_o = 0.25 \, Q \, t \, [z + sqrt \, (z^2 + 6 \, (x - x_o) \, / \, (Q \, t))] \qquad \textit{for } x > x_o \qquad (29)$$
$$= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad x \le x_o$$

This expression corresponds to *Equation (16)*, the only difference is that the constant 12 under square root is replaced by 6. The expression which corresponds to *Equation (17)* is obtained similarly by replacing the constant 4 under the square root by 2:

$$N_o = 0.25 \, Q \, t \, [z + sqrt \, (z^2 + 2 \, x \, / \, (Q \, t))] \qquad\qquad\qquad\qquad (30)$$

The overflow components of delay, stop rate and queue length at co-ordinated signals have the same form as those in *Equations (26) to (28)*, i.e. $(N_o \, x)$, $(N_o \, / \, qc)$ and (N_o), respectively, with N_o given by *Equation (29)*, or alternatively *Equation (30)*.

The uniform components (D_u, h_u, N_u) can be calculated using a deterministic simulation model such as the TRANSYT model. Simplified analytical expressions can also be used for this purpose, e.g. as in MITROP computer program (Gartner, et al 1975). It is recommended that the overflow components of stop rate and average queue length as described above are added to the stop rate and queue length calculations of the TRANSYT program. The correction to stops would allow for multiple stops under heavy traffic conditions, and would probably affect the optimum timing results.

7. CONCLUSION

The purpose of developing the time-dependent expressions given in this paper was to provide simple working models to predict traffic operating characteristics at signals, rather than obtain theoretically perfect but inevitably more complicated models. The expressions given in this paper are intended for use in a fashion similar to the use of the traditional Webster (1958) and Miller (1968) formulae which are limited to undersaturated cases.

It is possible to use the basic relationship given here to derive more comprehensive expressions to predict conditions in consecutive time periods. Such expressions will appear to be more complicated than those given in this paper, but can be implemented by means of a computer algorithm (e.g. see Mayne 1979).

A study of the optimum signal timings obtained using the time-dependent expressions given in this paper compared with those given by the Webster (1958) and Miller (1968) formulae is in progress.

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APPENDIX - A NOTE ON VARIOUS DELAY EXPRESSIONS

Ohno (1979) proposed the following steady-state expression for average delay at isolated fixed-time signals as a good approximation to an 'exact' theoretical solution obtained using a special computer algorithm for the' Poisson arrival-constant saturation headway' process.

$$d = 0.5\, c\, (1 - u)^2 / (1 - y) + ((1 - u) / (1 - y))\, (N_o / q) + \quad\quad\quad\quad\quad\text{(A.1)}$$
$$((1 - u) / (1 - y)) / (2\, s) + ((1 - u) / (1 - y)^2) / (2\, s)$$

where

d   =   average delay per vehicle in seconds,

u   = g / c = green time ratio, i.e. the proportion of the cycle that is effectively green

g   =   effective green time in seconds,

c   =   cycle time in seconds,

y   =   flow ratio ($=q/s$),

q   =   average arrival rate (flow) in vehicles per seconds,

s   =   saturation flow in vehicles per second,

N_o =   average overflow queue in vehicles obtained from *Equation (8)*.

The Miller (1968) formula for average delay, which is obtained from *Equation (9)* as $d = D / q$, is

$$d = 0.5\, c\, (1 - u)^2 / (1 - y) + ((1 - u) / (1 - y))\, (N_o / q) \quad\quad\quad\quad\quad\text{(A.2)}$$

where $x = (q\, c) / (s\, g)$ is the degree of saturation.

It is seen that the difference between *Equation* (A.1) and *Equation* (A.2) is merely due to the third and fourth terms; *Equation* (A.1) was called the modified Miller formula by Ohno.

The following average delay expression corresponds to the approximation given by *Equation (18)* used to modify the second term of *Equation* (9):

$$d = 0.5\, c\, (1 - u)^2 / (1 - y) + N_o\, x / q \qu\quad\quad\quad\quad\quad\quad\quad\text{(A.3)}$$

where x = degree of saturation ($= qc/sg$)

Also of interest is the widely used Webster (1958) formula:

$$d = 0.5\, c\, (1 - u)^2 / (1 - y) + 0.5\, x^2 / (q\, (1 - x)) \quad\quad\quad\quad\quad\text{(A.4)}$$
$$- 0.65\, (c / q^2)^{1/3}\, x^{(2+5u)}$$

Average delays calculated from the above expressions are given in *Tables A.1 and A.2* for the data used in Ohno's (1978) Figures 2 and 3. In *Table A.1*, c = 90 s, g = 45 s, s = 1.0 veh/s (3600 veh/h), hence u = 0.50, sg = 45. The value of y is varied and the average delay, d, is calculated for each y value. In Table A.2, c = 90 s, q = 0.4 veh/s (1400 veh/h), s = 1.0 veh/s, hence y = 0.40. The value of u is varied and the average delay is calculated. In both tables, the degree of saturation, x is also given as a better indicator of congestion levels. For *Equations* (A.1) and (A.2), N_o is calculated from *Equation (8)*. For *Equation* (A.3), N_o is calculated from *Equation (13)*.

The following conclusions are drawn from a comparison of delay values from expressions given in Tables *A.1 and A.2:*

(a)   The absolute differences are very small (about one second only in most cases). Accepting the results from *Equation (A.1)* as a basis for comparison, *Equations (A.2) and (A.3)* are found to give slightly lower values, whereas the Webster formula (*Equation A.4*) is found to give slightly higher values for high degree of saturation. However, the difference can hardly be said to be practically significant.

(b)   Ohno's (1978) work indicates some large *relative errors* for expressions of the type *Equations (A.2) to (A.4)*. However, consideration of the relative differences alone may be misleading since it is found that they increase as the absolute differences decrease. For example, Ohno's (1978) Figure 3 (for the data in *Table A.2* of this paper) shows very large errors for low degrees of saturation (high u values) in which case absolute delay values are of the order of 1 sec.

(c)   Approximations introduced in this paper, namely *Equation (13)* for the average overflow queue and *Equation (18)* for the random delay term are found to be satisfactory (compare the results from *Equations (A.2) and (A.3)* given in *Tables A.1 and A.2*) indicating that the basis of time-dependent expressions given in this paper are reliable.

Table A.1 - Average delay values (seconds) for c = 90 s, u = 0.50, s = 3600 veh/h

y =	0.10	0.20	0.30	0.40	0.45	0.47	
x =	0.20	0.40	0.60	0.80	0.90	0.94	
Eqn (A.1) *	13.1	14.8	17.0	20.4	25.5	32.1	Ohno (1978)
Eqn (A.2) *	12.5	14.1	16.1	19.3	24.2	30.7	Miller (1968)
Eqn (A.3) **	12.5	14.1	16.1	19.6	25.1	31.0	This paper
Eqn (A.4)	12.7	14.6	16.9	20.8	26.4	33.3	Webster (1958)

* N_o from Equation (8)
** N_o from Equation (13)

Table A.2 - Average delay values (seconds) for c = 90 s, u = 0.40, s = 3600 veh/h

u =	0.80	0.70	0.60	0.50	0.44	0.42	
x =	0.50	0.57	0.67	0.80	0.90	0.95	
Eqn (A.1) *	3.4	7.4	12.9	20.4	28.9	40.8	Ohno (1978)
Eqn (A.2) *	3.0	6.8	12.0	19.3	27.7	39.5	Miller (1968)
Eqn (A.3) **	3.0	6.8	12.0	19.6	28.7	39.8	This paper
Eqn (A.4)	3.5	7.5	13.0	20.8	29.8	42.4	Webster (1958)

* N_o from Equation (8)
** N_o from Equation (13)