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NOTE:

This paper is related to the intersection analysis methodology used in the SIDRA INTERSECTION software. Since the publication of this paper, many related aspects of the traffic model have been further developed in later versions of SIDRA INTERSECTION. Though some aspects of this paper may be outdated, this reprint is provided as a record of important aspects of the SIDRA INTERSECTION software, and in order to promote software assessment and further research.

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A GRAPHICAL EXPLANATION OF THE TWO PRINCIPLES AND TWO TECHNIQUES OF TRAFFIC ASSIGNMENT

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Abstract—The purpose of the paper is to clarify issues related to the two fundamental principles of traffic assignment and two techniques used for its solution. Firstly, the inter-relationships among the average, marginal, total and integral forms of the travel time (cost)–flow function are discussed. The choice of the appropriate form of the travel time (cost)–flow function depends on the assignment technique (heuristic loading or mathematical optimisation) and the assignment principle (user-optimising or system-optimising) adopted for use. A table is given as an aid for this choice. A graphical explanation is then presented, based on the fundamental two-route single-demand assignment problem to illustrate the usefulness of the graphical approach for simple analysis of various issues related to traffic assignment.

1. INTRODUCTION

The procedural equivalency of the user-optimising and system-optimising assignment principles (Wardrop, 1952; Dafermos and Sparrow, 1969; Potts and Oliver, 1972; Ruiter, 1974; Steenbrink, 1974a; Florian, 1976; Wigan, 1977) makes it possible to use either a mathematical optimisation technique or a heuristic loading technique to satisfy either of these two traffic assignment principles by choosing the appropriate form of the travel time (cost)–flow function. Although this characteristic of the traffic assignment problem has been described and examined in theoretical terms in the literature, its importance has not been fully realised by the profession and only limited use of it has been made in traffic assignment related work.

The properties of a hyperbolic (Mosher, 1963; Davidson, 1966), a polynomial (U.S. Dept. of Transportation, 1976) and an exponential (Wigan and Luk, 1976) travel time (cost)–flow function have been discussed with respect to their ability to predict queueing effects in a recent paper (Akcelik, 1978b). It has been suggested that both the hyperbolic and polynomial functions predict queueing effects reasonably well. It has also been suggested that the use of varying values of a quality-of-service parameter in a particular travel time-flow function to represent different types of road is important in the traffic assignment process.

Once a particular function is chosen and the values of quality-of-service and other parameters (capacity, free-flow travel time) are established, it is necessary to select the most appropriate *form* of that function. This depends on both the assignment technique (mathematical optimisation or heuristic loading) and the assignment principle (user-optimising or system-optimising) adopted for use. This paper discusses the average, marginal, total and integral forms of the travel time (cost)–flow function, and presents a table in the form of a matrix to be used for selecting the appropriate function form. A graphical explanation is then presented, based on the fundamental two-route single-demand assignment problem. This graphical approach can be used for simple analysis of traffic assignment issues, for example, the effects due to the difference between perceived and actual travel times (costs), road pricing, toll and route control (Akcelik, 1977).

2. TRAFFIC ASSIGNMENT: TWO PRINCIPLES AND TWO TECHNIQUES

The two fundamental traffic assignment principles first enunciated by Wardrop (1952) can be summarised as follows:

2.1 *User-optimising assignment*

This principle is based on the assumption that an individual driver chooses the route that has the smallest travel time (cost) between his origin and destination. According to this principle, no single driver can reduce his own travel time by choosing an alternative route under equilibrium conditions and the resulting flow pattern is such that the travel times (average) on all routes actually used are equal and less than the travel time on any unused route, that is

$$\begin{aligned} t_1 = t_2 = \cdots = t_m = t_u \le t_{m+1} \le t_{m+2} \le \cdots \le t_n \\ q_r > 0, \qquad r = 1, 2, \ldots, m, \\ q_r = 0, \qquad r = m+1, m+2, \ldots, n \end{aligned} \tag{1}$$

where r = route number, n = number of available routes, m = number of routes actually used, q_r = part of the total O–D flow which uses route r, t_r = travel time along route r (sum of link travel times) and t_u = equilibrium (average) travel time.

2.2 *System-optimising assignment*

This principle is based on the assumption that the total travel time (cost) over the entire network is minimised. The total system travel time is given by

$$P = \sum_{i=1}^{N} T_i \tag{2}$$

where $T_i = q_i t_i$ = total travel time (sum of all drivers' travel times) along link i, q_i = flow on link i, t_i = average travel time along link i and N = number of links in the network. The system optimising flow pattern is given by the minimum value of P.

Traditionally,

(i) heuristic assignment techniques (capacity-restraint type, incremental or total loading methods) have been employed for finding a user-optimising flow pattern, which will be referred to as A.T.1 below, and

(ii) optimisation (mathematical programming) techniques have been used for computing a system-optimising flow pattern, which will be referred to as A.T.2 below.

More recently, (a) heuristic assignment techniques (A.T.1) have been used to calculate system-optimising flow patterns (Akcelik, 1974, 1978a; Steenbrink, 1974a, b), and (b) a mathematical optimisation technique (A.T.2) usually referred to as an equilibrium assignment has been developed to compute user-optimising flow patterns (Nguyen, 1974; LeBlanc, 1975; LeBlanc *et al.*, 1975; Florian, 1976; Wigan and Luk, 1976; Wigan, 1977; Daganzo, 1977). These new approaches make use of the procedural equivalency of the user- and system-optimising assignment principles which is discussed below.

(a) The use of heuristic loading techniques (A.T.1) for system-optimising assignment purposes is based on the fact that, in a system-optimising flow pattern, "the marginal travel times (costs) on all routes actually used are equal and less than that on any unused route" (e.g. see Akcelik, 1977). The marginal travel time function can be derived from

$$\tau = \frac{dT}{dq} = t + \frac{dt}{dq}q = t + \sigma q \tag{3}$$

where $T = tq$ = total travel time (e.g. in vehicle-hours per hour) and $\sigma = dt/dq$ = rate of change in average travel time with increasing flow, and the system-optimising flow pattern corresponds to

$$\begin{aligned} &\tau_1 = \tau_2 = \cdots = \tau_m = \tau_s \le \tau_{m+1} \le \tau_{m+2} \le \cdots \le \tau_n \\ &q_r > 0, \qquad r = 1, 2, \ldots, m, \\ &q_r = 0, \qquad r = m+1, m+2, \ldots, n \end{aligned} \tag{4}$$

where r, n, m, q_r are defined as for eqn (1), τ_r = marginal travel time along route r and τ_s = equilibrium (marginal) travel time.

It is because of this property that the heuristic iterative techniques (A.T.1), which try to find equilibrium conditions where travel times on alternative routes are equal, can be used to compute: (i) a user-optimising flow pattern by employing the *average* travel time-flow function, $t(q)$, or (ii) a system-optimising flow pattern by employing the *marginal* travel time-flow function $\tau(q)$.

(b) The use of a mathematical optimisation technique (A.T.2) for user-optimising assignment purposes is based on the use of an integral function derived from the average travel time (cost) function. The integral function is given by

$$\theta = \int_0^q t(y)\,dy \tag{5}$$

and it is defined in such a way that its marginal travel time is the actual average travel time, that is

$$\tau^* = \frac{d\theta}{dq} = t. \tag{6}$$

The total system travel time using total link travel times from eqn (5) is

$$P^* = \sum_{i=1}^{N} \theta_i = \sum_{i=1}^{N} \left[\int_0^{q_i} t_i(y)\,dy \right] \tag{7}$$

and the minimum value of P^* gives a user-optimising assignment.

This is because it leads to an equal "marginal" travel time assignment where the marginal travel time is the average travel time as indicated by eqn (6). Hence, an assignment technique which employs a mathematical optimisation algorithm (A.T.2) can be used to compute: (i) a user-optimising flow pattern by utilising $\theta(q)$ as the total link travel time function, or (ii) a system-optimising flow pattern by employing $T(q)$ as the total link travel time function.

In summary, a user-optimising or a system-optimising assignment solution can be obtained using either a heuristic loading (A.T.1) or a mathematical optimisation (A.T.2) technique by choosing one of the average (t), marginal (τ), total (T) and integral (θ) forms of the travel time (or cost) function. The choice of the appropriate function form which achieves the desired assignment solution can be obtained from Table 1.

The average, marginal, total and integral forms of the hyperbolic (Mosher, 1967; Davidson, 1966), polynomial (U.S. Dept. of Transportation, 1976) and exponential travel time functions (Wigan and Luk, 1976) are given in Table 2, where t_f = free-flow travel time and x degree of saturation, i.e. the ratio of flow to capacity, q/k. The parameters m, β and b of hyperbolic, polynomial and exponential functions, respectively, are quality-of-service or delay parameters. These parameters allow for a family of curves to be developed from the same function to represent different road types and environmental conditions (e.g. uninterrupted, interrupted, major road, minor road, etc.).

A comparison of the three functions given in Table 2 with respect to their ability to predict queueing effects has been described in a recent paper (Akcelik, 1978b). It has been found that the hyperbolic and polynomial functions give a reasonable representation of the queueing effects. An advantage of the hyperbolic function is that there are experimental values available for its quality-of-service parameter (Akcelik, 1978b).

Table 1. The choice of the appropriate form of travel time function for user-optimising and system-optimising assignment

	User-optimising assignment	System-optimising assignment
Heuristic loading technique (A.T.1)	Average travel time, $t(q)$ $t_1 = t_2 = \cdots = t_m$	Marginal travel time, $\tau(q)$ $\tau_1 = \tau_2 = \cdots = \tau_m$
Mathematical optimisation technique (A.T.2)	Integral travel time, $\theta(q)$ $(\Sigma\theta_i)_{\min}$	Total travel time, $T(q)$ $(\Sigma T_i)_{\min}$

Table 2. Normalized average, marginal, total and integral forms of three travel time functions

Function	Average travel time $t/t_f =$	Marginal travel time $\tau/t_f =$	Total travel time $T/kt_f =$	Integral travel time $\theta/kt_f =$
Hyperbolic	$\frac{1-mx}{1-x}$	$m+\frac{1-m}{(1-x)^2}$	$\frac{x(1-mx)}{(1-x)}$	$mx-(1-m) \times \ln(1-x)$
Polynomial	$1+\beta x^4$	$1+5\beta x^4$	$x(1+\beta x^4)$	$x\left(1+\frac{\beta}{5}x^4\right)$
Exponential	e^{bx}	$(1+bx)\,e^{bx}$	$x\,e^{bx}$	$\frac{1}{b}(e^{bx}-1)$

3. A GRAPHICAL EXPLANATION

In order to develop a graphical explanation of the procedural equivalency of the user-optimising and system-optimising assignment principles discussed above, Davidson's hyperbolic travel time function has been used. The meanings of average, marginal, total and integral travel time functions are illustrated in Fig. 1 (drawn for $m = 0.60$ in the hyperbolic function). The diagrams which are related to eqns (3), (5) and (6) are self-explanatory. The differences between $t(q)$ and $\tau(q)$ curves in Fig. 1(a) and $T(q)$ and $\theta(q)$ curves in Fig. 1(b) indicate the potential for a difference between user- and system-optimising assignment patterns. It is seen that differences get larger as flows approach capacity hence substantial differences may exist in networks near saturation. The difference in the behaviour of the $T(q)$ and $\theta(q)$ curves observed in Fig. 1(b) indicates that a correct prediction of the link travel times in the near-saturated flow region is important. Therefore the choice of a travel time-flow function which simulates such queueing effects is important. It is also seen that the integral travel times are rather insensitive to congestion effects and this implies that individual users may be ready to accept delays due to queueing (user-optimising equilibrium) whereas the actual total travel times are very sensitive to congestion (system-optimising equilibrium). It should be noted that these conclusions are not dependent on a particular travel time-flow relationship, i.e. the hyperbolic function shown in Fig. 1. For example, graphs for the equivalent polynomial function ($\beta = 5.9$ as discussed in Akcelik, 1978b) indicate similar patterns. Any function (model) which can predict queueing effects under near-congested conditions would possess the equilibrium characteristics discussed in this paper.

A comparison of user- and system-optimising equilibrium patterns may be explained graphically for the fundamental traffic assignment case. A single origin-destination (O–D) network with a pair of distinct routes is considered, which is illustrated in Fig. 2. The O–D flow, Q is subject to distribution over routes 1 and 2 which connect the given O–D pair. There are basic flows q_{b1} and q_{b2} on these routes from other O–D pairs. These are considered to be constant flows which exist on these routes, that is they are not subject to the assignment but they affect the assignment process by contributing to travel times on alternative routes. An assignment to the network will result in a flow pattern (q_1, q_2). If the proportion of Q using route 1 is f, then the route flows are given by

$$q_1 = q_{b1} + fQ$$
$$q_2 = q_{b2} + (1-f)Q \tag{8}$$

where $0 \le f \le 1.0$. Therefore the network flow, $\bar{Q}$ is equal to the sum of the O–D flow and basic flows, that is

$$\bar{Q} = Q + q_{b1} + q_{b2} = q_1 + q_2. \tag{9}$$

The solution to the fundamental problem is illustrated in Fig. 3. In this diagram, travel time curves are shown for each route with the q_1 and q_2 axes drawn in opposite directions and to obtain $q_1 + q_2 = \bar{Q}$ at each point. Graphs in Fig. 3 have been shown for the numerical

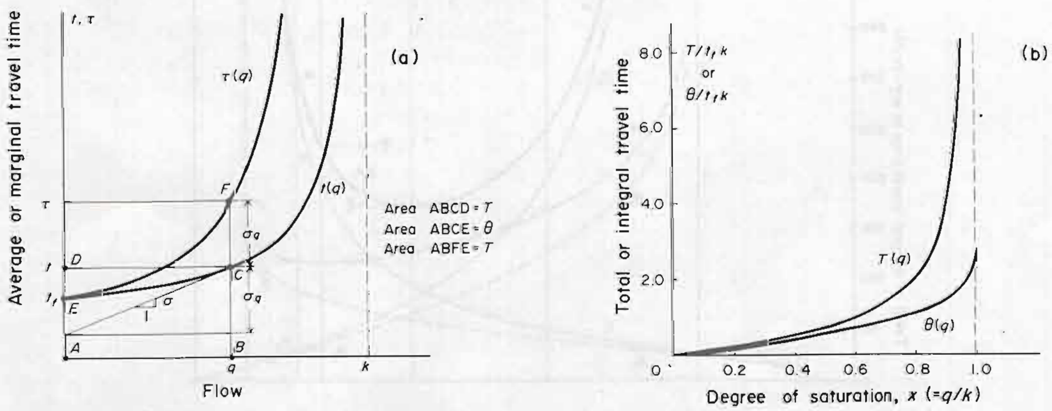

Fig. 1. Average, marginal, total and integral forms of a travel time-flow function.

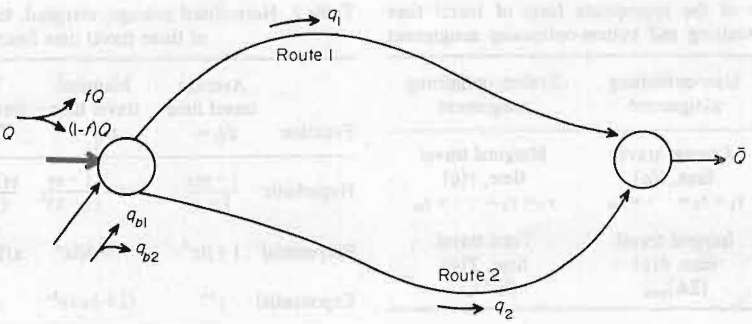

Fig. 2. Single origin-destination network with a pair of distinct routes.

example described in Table 3 using the hyperbolic function. The parameters of the travel time function have been chosen in such a way that Route 1 is the major route and Route 2 is the minor route. An O–D flow of 1000 veh/hr is subject to assignment, therefore the network flow $\bar{Q} = 3000$ veh/hr. The solutions are given in Table 4. It should be noted that, as far as the user-optimising and system-optimising solutions are concerned, this problem is equivalent to one with zero basic flows and an O–D flow of 3000 veh/hr.

It is seen that the user-optimising solution, given by $t_1 = t_2$ (Fig. 3a) corresponds to the minimum value of P^* (Fig. 3b) which is defined by eqn (7). Similarly, the system-optimising solution given by the minimum value of P (Fig. 3b) which is defined by eqn (2), corresponds to $\tau_1 = \tau_2$ (Fig. 3a).

4. DISCUSSION

Of particular interest is the difference between the user-optimising and system-optimising assignment pat-

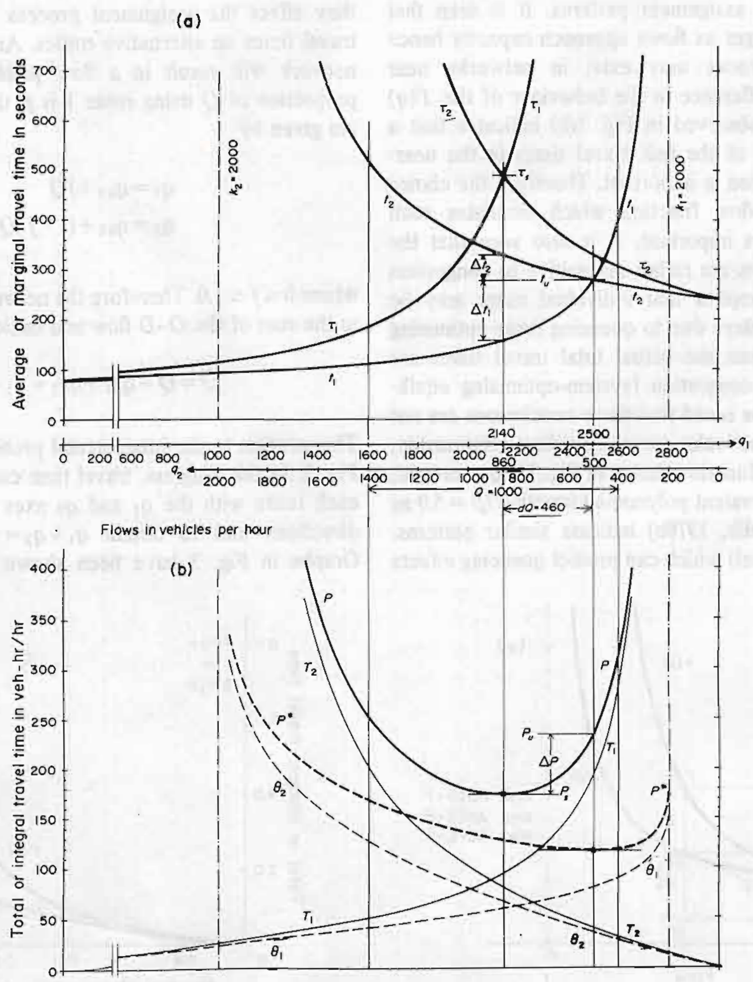

Fig. 3. Fundamental diagram of traffic assignment.

Table 3. Data for numerical example

Route	Free-flow travel time t_f (sec)	Capacity k (veh/hr)	Quality of service parameter, m	Basic flow q_b (veh/hr)
1	80	2800	0.70	1600
2	240	2000	0.50	400

terns in terms of both total network and individual vehicle travel times. A user-optimising assignment pattern would result from self-assignment of individual vehicles based on the choice of the quickest (or cheapest) route whereas a system-optimising pattern could be achieved by some measure of route control (i.e. control over the route choice decisions of individual drivers). In the above example, the difference in system performance between the two patterns is $\Delta P =$ 60.55 veh-hr/hr (26%). This is the amount of savings which could be achieved by diverting $dQ = 460$ veh/hr from Route 1 to Route 2, and the resulting increase in the travel time of vehicles diverted to Route 2 would be $\Delta t_2 = 51$ sec (18%). It should be noted that it may also be possible to decrease the travel time of vehicles diverted to the minor route by means of various traffic control measures. This can be solved as a combined traffic assignment-control problem, which has been illustrated in a previous paper by the author (Akcelik, 1977).

The differences in individual vehicle and system performance discussed above correspond to perfect (theoretical) user- and system-optimising patterns. In practice, drivers choose their routes according to perceived travel times (costs) rather than actual travel times (costs). Depending on the way perceived travel times differ from actual travel times, the real-life flow pattern will differ from the theoretical user-optimising equilibrium pattern. Let us assume that the perceived travel time-flow relationship $t_p(q)$ has the same form as the actual travel time-flow relationship $t(q)$. If drivers do not perceive congestion delay to its real extent, then $t_p < t$, and the resulting equilibrium pattern will give a system performance which is worse than that in the theoretical user-optimising pattern, i.e. $P > P_u$. If drivers perceive travel times in such a way that $\tau > t_p > t$, then the resulting flow pattern will lie between the user- and system-optimising patterns with a system performance $P_s < P < P_u$. Hence, depending on drivers' perception of travel times, benefits from perfect route controls are either smaller or greater than the theoretical value. On the other hand, it would be difficult to achieve perfect route control in practice and near-optimal solutions should be acceptable.

5. CONCLUSION

The fundamental diagram of traffic assignment presented in this paper (Fig. 3) can be used for simple analysis of various issues related to traffic assignment. This approach was used by the author for the basic analysis of a combined traffic assignment/control problem (Akcelik, 1977). The effects of differences between perceived and actual travel times (costs) discussed in the previous section can be easily demonstrated with the aid of the fundamental diagram. Similarly, it can be used for other purposes, for example: (i) to define an equilibrium region instead of an equilibrium point for user-optimising assignment by developing travel time (cost) curves $(t + \Delta t)$ and $(t - \Delta t)$ for each route, where Δt defines the range of variation in perceived travel times (e.g. 20% of t), and (ii) to study road pricing and toll effects, e.g. by considering a major/minor route situation and adding a toll charge to the major route travel time (cost) so that the resulting equilibrium is forced to a system-optimising pattern.

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Table 4. Solution

	Route flows (veh/hr)		Degrees of saturation		Average route travel times (sec)		Marginal route travel times (sec)		Total system travel time (veh-hr/hr)	Integral system travel time (veh-hr/hr)
	q_1	q_2	x_1	x_2	t_1	t_2	τ_1	τ_2	P	P^*
User-optimising flow pattern	2500	500	0.89	0.25	$t_u = 280$		2146	333	233.33	116.43
System-optimising flow pattern	2140	860	0.76	0.43	158	331	$\tau_s = 489$		172.78	126.42

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